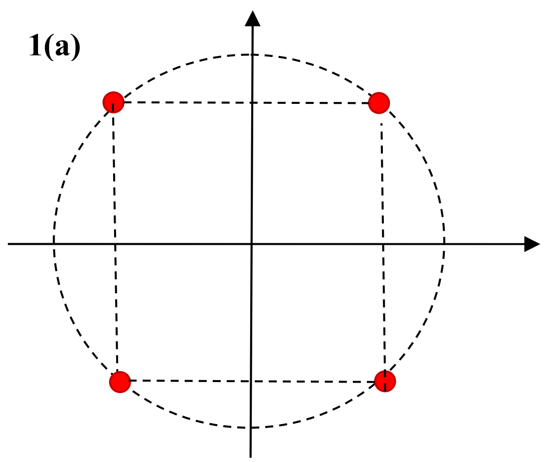
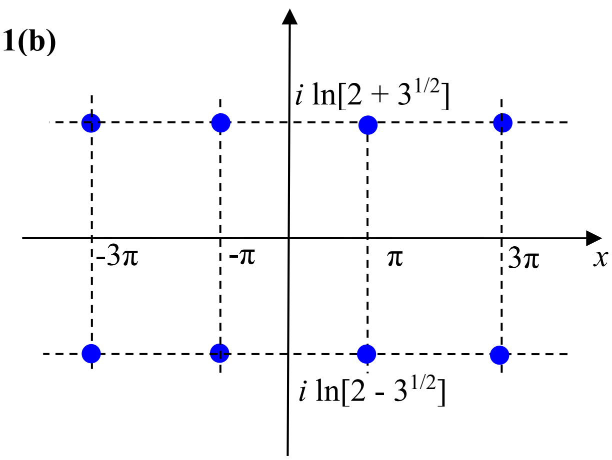
Math 656 • Midterm Exam • March 8, 2016 • Victor Matveev

1) (15pts) Find all values of z in polar or Cartesian form, and plot them as points in the complex plane:

$$(a)\quad z=\left(1-i\sqrt{3}\right)^{3/4}=\left(2e^{-i\frac{\pi}{3}+i2\pi n}\right)^{3/4}=\sqrt[4]{8}\,e^{-i\frac{\pi}{4}+i\frac{3\pi n}{2}}=\begin{cases} n=0: & \sqrt[4]{8}\,e^{-i\frac{\pi}{4}} \\ n=1: & \sqrt[4]{8}\,e^{-i\frac{\pi}{4}+i\frac{3\pi}{2}}=\sqrt[4]{8}\,e^{i\frac{5\pi}{4}}=\sqrt[4]{8}\,e^{-i\frac{3\pi}{4}} \\ n=2: & \sqrt[4]{8}\,e^{-i\frac{\pi}{4}+i\frac{6\pi}{2}}=\sqrt[4]{8}\,e^{i\frac{11\pi}{4}}=\sqrt[4]{8}\,e^{i\frac{3\pi}{4}} \\ n=-1: & \sqrt[4]{8}\,e^{-i\frac{\pi}{4}-i\frac{3\pi}{2}}=\sqrt[4]{8}\,e^{-i\frac{7\pi}{4}}=\sqrt[4]{8}\,e^{+i\frac{\pi}{4}} \end{cases}$$





$$(b)\quad \cos z=-2 \;\Rightarrow\; \frac{e^{iz}+e^{-iz}}{2}=-2 \;\Rightarrow\; e^{iz}+e^{-iz}+4=0 \,\Big|\times e^{iz} \;\Rightarrow\; \underbrace{e^{i2z}}_{s^2}+4\underbrace{e^{iz}}_{s}+1=0$$

$$\Rightarrow s^2+4s+1=0 \Rightarrow s_{1,2}=\frac{-4+(16-4)^{1/2}}{2}=-2\pm\sqrt{3} \quad\Rightarrow\quad z=-i\log(-2\pm\sqrt{3})$$

Note that $-2\pm\sqrt{3}<0$ so $-2\pm\sqrt{3}=-\left(2\pm\sqrt{3}\right)=e^{i\pi}\left(2\pm\sqrt{3}\right)$

$$z=\begin{cases}-i\log\left(\left(2+\sqrt{3}\right)e^{i\pi}\right)=-i\left(\ln\left(2+\sqrt{3}\right)+i\left(\pi+2\pi n\right)\right)\\ -i\log\left(\left(2-\sqrt{3}\right)e^{i\pi}\right)=-i\left(\ln\left(2-\sqrt{3}\right)+i\left(\pi+2\pi n\right)\right)\end{cases}\Rightarrow \boxed{z=\cos^{-1}2=\pm i\ln\left(2+\sqrt{3}\right)+\pi\left(1+2n\right)\quad n\in\mathbb{Z}}$$

In the last step we used the fact that $\ln\left(2-\sqrt{3}\right)=-\ln\left(2+\sqrt{3}\right)$ because $\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)=1$

2) (15pts) Sketch the image of the region $\{z \in \mathbb{C} : 1 \le |z| \le e,\ \operatorname{Im} z \ge 0\}$ under the mapping $w = i\,\mathrm{Log}(iz)$. You may consider this transform as a sequence of 3 separate, simple steps. Hint: use polar form for the original variable *z*, and note the slight complication from the fact that Log(z) is the branch with $\arg z \in (-\pi, \pi]$

Step 1:

$z \to \tilde{z} = iz$: rotation by $\frac{\pi}{2}$

"Half-ring" in the upper half-plane $\Rightarrow$ "Half-ring" in the right half-plane

$$\{1 \le |z| \le e,\ 0 \le \arg z \le \pi\} \Rightarrow \left\{1 \le |\tilde{z}| \le e,\ \arg \tilde{z} \in \left[\frac{\pi}{2}, \pi\right] \cup \left[-\pi, -\frac{\pi}{2}\right]\right\}$$ Note the branch cut!

Step 2:

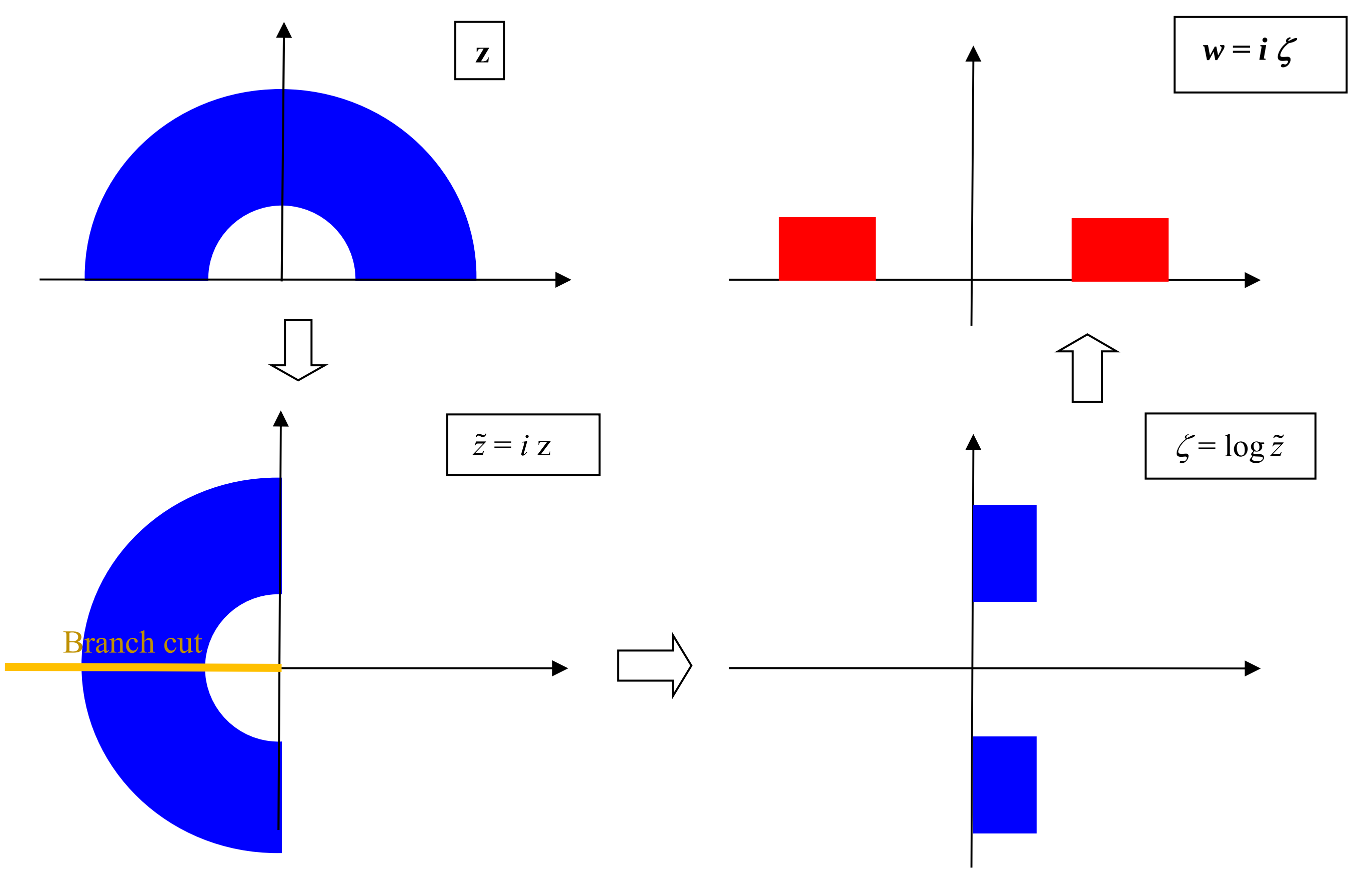
$\tilde{z} \to \zeta = \mathrm{Log}(\tilde{z})$:

"Half-ring" in the right half-plane $\Rightarrow$ Two rectangles (would be 1 rectangle if not for the branch cut!)

$$\left\{1 \le |\tilde{z}| \le e,\ \arg \tilde{z} \in \left[\frac{\pi}{2}, \pi\right] \cup \left[-\pi, -\frac{\pi}{2}\right]\right\} \Rightarrow \left\{0 \le \operatorname{Re}\zeta \le 1,\ \operatorname{Im}\zeta \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]\right\}$$

Step 3: $\zeta \to w = i\zeta$: rotate two rectangles by $\frac{\pi}{2}$:

$$\left\{0 \le \operatorname{Re}\zeta \le 1,\ \operatorname{Im}\zeta \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right]\right\} \Rightarrow \boxed{\left\{\operatorname{Re} w \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \pi\right],\ 0 \le \operatorname{Im} w \le 1\right\}}$$



3) (25pts) Calculate each integral over the indicated circle, or explain why the integral equals zero:

a) $\oint_{|z|=3} \frac{dz}{\left(e^z+1\right)^9} = 0$ by $\boxed{\text{C.G.T.}}$ since the nearest singularity is at $z = i\pi$, outside the contour

b) $\oint_{|z|=5} \frac{e^z\, dz}{\left(e^z+1\right)^9} = 0$ by $\boxed{\text{F.T.C.}}$ since the anti-derivative $F(z) = -\frac{1}{8\left(e^z+1\right)^8}$ exists on entire contour

c) $\oint_{|z|=2} \frac{\sin\left(z^2\right) dz}{z^2-2i\,z-1} = \oint_{|z|=2} \frac{\sin\left(z^2\right) dz}{\left(z-i\right)^2} = 2\pi i \frac{d}{dz}\sin\left(z^2\right)\Big|_{z=i} = 2\pi i\left(2i\right)\cos\left(-1\right) = \boxed{-4\pi\cos 1}$

d) $\oint_{|z|=R} \frac{dz}{\sqrt{z}} = \underbrace{2\sqrt{z}\Big|_{Re^{-i\pi}}^{Re^{i\pi}}}_{\text{Jump across branch cut}} = 2\sqrt{R}\left[e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}\right] = \boxed{4i\sqrt{R}}$ Equivalently, can obtain this by parametrizing $z=Re^{i\theta}$

e) $\int_{|z|=R} \bar{z}\, dz = \int_0^{2\pi} R\,e^{-i\theta}\left(iR\,e^{i\theta}d\theta\right) = iR^2\int_0^{2\pi} d\theta = \boxed{i\,2\pi R^2}$

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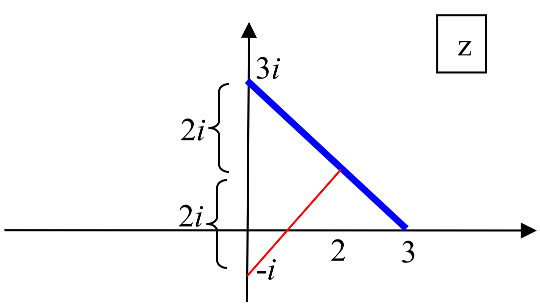
4) (15pts) Find the bound on $\left|\int_C \frac{\cosh z}{z^2+2iz-1}dz\right|$, where the integration contour C is a straight line connecting points $z=3i$ and $z=3$. Hint: express cosh z in terms of exponentials.

$$\left|\int_C \frac{\cosh z}{z^2+2iz-1}dz\right| = \left|\int_C \frac{\left(e^z-e^{-z}\right)/2}{\left(z+i\right)^2}dz\right| \le \int_C \frac{\left(\left|e^z\right|+\left|e^{-z}\right|\right)/2}{\left|z+i\right|^2}\left|dz\right| \le \frac{\max\left(\frac{e^x+e^{-x}}{2}\right)}{\min\left|z+i\right|^2}\underbrace{3\sqrt{2}}_{L}$$

Here $\min_{Line}\left|z+i\right|^2$ is the shortest squared distance from $-i$ to the line from $3i$ to 3

Shortest distance is shown in the Figure: $\min_{Line}\left|z+i\right|^2 = 2^2+2^2 = 8 \Rightarrow \boxed{\left|\int_C \frac{\cosh z}{z^2+2iz-1}dz\right| \le \frac{3\sqrt{2}\cosh 3}{8}}$



============================ **Pick 2 problems between 5, 6, 7** ============================

5) (15pts) Consider *any* branch of function $f(z)=\left(\frac{z}{z-1}\right)^{1/2}$, describe its branch cut(s) and describe the jump discontinuity of this function across the branch cut(s). Finally, use this branch to compute $f(i)$

$$f(z)=\left(\frac{z}{z-1}\right)^{1/2} \Rightarrow \frac{z^{1/2}}{(z-1)^{1/2}}=\frac{\sqrt{r_1}e^{i\theta_1/2}}{\sqrt{r_2}e^{i\theta_2/2}}=\sqrt{\frac{r_1}{r_2}}e^{i\frac{\theta_1-\theta_2}{2}}$$

Let's choose the branch defined by $\begin{cases} z \equiv r_1\exp(i\theta_1) \\ z-1\equiv r_2\exp(i\theta_2)\end{cases} \quad -\pi\le\theta_{1,2,3}<\pi$

Compute $f(i)$ for this prescription: $\begin{cases} r_1=1, & \theta_1=\pi/2 \\ r_2=\sqrt{2}, & \theta_2=3\pi/4\end{cases} \Rightarrow f(i)=\sqrt{\frac{1}{\sqrt{2}}}e^{i\frac{\frac{\pi}{2}-\frac{3\pi}{4}}{2}}=\boxed{\frac{e^{-i\pi/8}}{\sqrt[4]{2}}}=\boxed{\frac{\sqrt{\sqrt{2}+1}}{2}-i\frac{\sqrt{\sqrt{2}-1}}{2}}$

This branch of $f(z)$ has a cut on the real axis $x\in(0,1)$:

- $x\in(-\infty,0)$: $\theta_{1,2}$ both jump by $2\pi \Rightarrow \theta_1-\theta_2$ is continuous $\Rightarrow$ **no cut**
- $x\in(0,\ 1)$: θ_2 jumps by $2\pi \Rightarrow f(z)$ acquires jump factor $\sqrt{\frac{r_1}{r_2}}\exp\left(-i\frac{2\pi}{2}\right)=-\sqrt{\frac{r_1}{r_2}}$ } **branch cut**
- $x\in(1,+\infty)$: $\theta_{1,2}$ are continuous $\Rightarrow$ **no cut**

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6) (15pts) Can the function $f(z)=e^{i\theta(z)}$ be analytic anywhere in domain *D* if $\theta(z)$ is a real non-constant function in *D*? Use any method or theorem you like to answer this question.

Can't be analytic anywhere (apart from any open subset of *D* where $\theta(z)=const$). Two ways to prove this:

1. Method 1: Cauchy-Riemann equations $\frac{\partial f}{\partial x}\overset{?}{=}-i\frac{\partial f}{\partial y} \Rightarrow \left\{\begin{aligned} \frac{\partial f}{\partial x}&=ie^{i\theta}\frac{\partial\theta}{\partial x} \\ -i\frac{\partial f}{\partial y}&=-i\cdot ie^{i\theta}\frac{\partial\theta}{\partial y}=e^{i\theta}\frac{\partial\theta}{\partial y}\end{aligned}\right\} \Rightarrow \boxed{\underbrace{i\frac{\partial\theta}{\partial x}}_{\text{Imaginary}}\overset{?}{=}\underbrace{\frac{\partial\theta}{\partial y}}_{\text{Real}}}$

   This equality obviously can't be satisfied unless both derivatives are zero, which corresponds to constant θ.

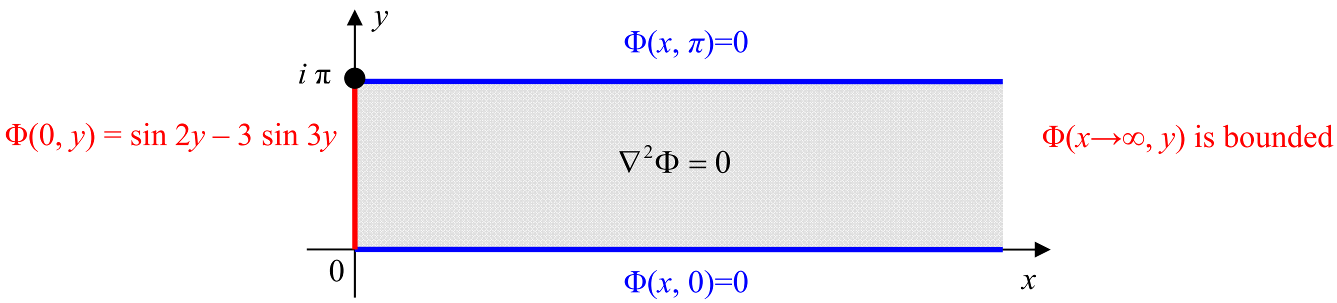
2. Method 2: Max Modulus Principle, but with an extra step, since Max Modulus Principle doesn't rule out the possibility that $|f|=const$ for non-constant Re(*f*) and Im(*f*) [*no points subtracted if you didn't do this step*]:

   $|f|^2=e^{i\theta(z)}e^{-i\theta(z)}=1=const;\ |f|^2=u^2+v^2$. Let's prove that *u* and *v* are also constant, and thus *θ*=const:

   Maximum of u^2 is achieved on the boundary (proven in the homework), but for non-constant *u* this would correspond to the minimum of v^2, which contradicts the Maximum / Minimum Principle for harmonic functions proven in the homework. Therefore, constant $|f|$ is only possible if both *u* and *v* are constant.

Finally, note that the Liouville Theorem *is not applicable* here, since it only concerns the case $D=\mathbb{C}$

7) (15 pts) Solve the boundary value problem for the Laplace's equation $\nabla^2\Phi = 0$ in an infinite strip, with boundary conditions indicated below (Φ is a real function). Hint: consider analytic functions of form $f(z) = Ae^{kz}$, where *A* and *k* are real constants. Make sure to satisfy all four boundary conditions!



Solution is obvious: pick negative k $(k = -2$ and $k = -3)$ to ensure that the solution is bounded at $x \to \infty$, and use imaginary part of the analytic function as the solution: $\Phi = \text{Im}\left[A_1 e^{-2z} + A_2 e^{-3z}\right]$

Boundary condition on the left gives A_1 and A_2:

$$\boxed{\begin{array}{l} k = -2: A_1 = -1 \\ k = -3: A_1 = +3 \end{array}} \Rightarrow \Phi(x, y) = \text{Im}\left[-e^{-2z} + 3e^{-3z}\right] = -e^{-2x}\sin(-2y) + 3e^{-3x}\sin(-3y) = \boxed{e^{-2x}\sin(2y) - 3e^{-3x}\sin(3y)}$$